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Higher associate class partially balanced incomplete block designs using graphs for agricultural experiments

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Abstract
Some new association schemes and construction methods of 3- and 4-associate class partially balanced incomplete block (PBIB) designs based on these schemes using chosen graphs have been given. A catalogue is included of efficient PBIB designs with number of treatments (v) ≤ 100.

Keywords: Pappus hexagon association scheme, partially balanced incomplete block design, star polygon association scheme, extended G_6 graph association scheme

1. Introduction
Partially balanced incomplete block (PBIB) designs are formally introduced by Bose and Nair (1939) [3]. Two class PBIB designs have been extensively studied in the literature and an exhaustive catalogue of these designs found in Clatworthy (1973) [4]. A lot of literature is available on PBIB designs based on 3- or higher class association schemes. PBIB (3) designs based on rectangular association scheme are an important class of block designs with factorial structure for experiments with two factors were studied by Vartak (1955) [34]; Sharma and Das (1985) [24]; Suen (1989) [29]; Srivastava *et al.* (2000) [28]; and Parsad *et al.* (2007a, 2007b) [18, 19]. The nested group divisible designs, a class of PBIB (3) designs, useful for 3-factor experiments was introduced by Roy (1953) [22] were subsequently studied by Raghavarao (1960) [20]; Miao *et al.* (1996) [15]; and Kageyama and Singh (2002) [10]. More generalized association scheme called extended group divisible association scheme and designs based on this scheme are known as extended group divisible (EGD) designs was discovered by Hinkelmann (1964) [9]. Many useful applications of these designs and their catalogue are given in Parsad *et al.* (2007a, 2007b) [18, 19]. PBIB (3) designs using circular lattices were developed by Rao (1956) [21] for $v = 2n^2$ treatments, where $n \geq 2$ and these were further generalized by Varghese and Sharma (2004) [33] to accommodate $2sn^2$ treatments for $n, s \geq 2$. Also, Varghese *et al.* (2004) [32] reported exhaustive work on 3-class PBIB designs with a comprehensive catalogue for $v, b \leq 100$, $k \leq 20$ and their applications to partial diallel crosses. Sharma *et al.* (2010) [25] extended the work on PBIB (3) designs further by proposing tetrahedral and cubical association schemes and methods of constructions of PBIB (3) designs based on these schemes. Garg *et al*. (2011) [7] developed some new triangular and four associate class PBIB designs in two replications. Furthermore, Sharma *et al.* (2013) [23] presented web solutions for PBIB designs. Garg and Farooq (2014) [6] introduced 2- and 3-associate class PBIB designs through chosen lines and graphs. Kipkemoi *et al*. (2013) [13] and Kipkemoi *et al*. (2015) [14] also gave some PBIB (3) designs in two replications. Recently, Singh and Garg (2020) [26] constructed some 2- and 3-associate class PBIB designs through edges and paths of graphs. Investigations of association schemes for 4-associate class PBIB designs have been curbed mainly to Nair (1951) [16]; Tharthare (1963, 1965) [31, 30]; and Garg *et al*. (2011) [7].
In the present investigation, three new association schemes with three and four associate classes and methods for constructing PBIB designs based on these schemes using some graphs are given in Section 2. Conclusions and future scope are covered in Section 3. Catalogues of PBIB designs for $v \leq 100$ along with the coefficient of error variance in the average variance $(\bar{V})$ and efficiency (E) as compared to a randomized complete block design have been obtained from these construction methods and are presented in the Appendix.

2. Graphical Association schemes and Constructions of PBIB designs
We present 3- and 4-class graphical association schemes which may be considered as generalizations of Theorem 3.1, Theorem 3.2, Theorem 3.3, and Theorem 3.6 of Garg and Farooq (2014) [6] and methods of constructing PBIB designs based on these schemes in the sequel.

2.1 Star Polygon Association scheme

Let $v = 10m$ $(m \geq 2)$ be the number of treatments. Arrange these treatments on the vertices of a Star Polygon graph such that each vertex contains exactly m distinct treatments. Now we define the association scheme on these v treatments as follows: Treatment β is the first associate of α, if β lies on the same vertex of α; the second associate, if β lies on any of the quadruplets (where each quadruplet has precisely four equi-distant vertices) that intersect the vertex of α and third associates, otherwise. The parameters of first kind and association matrices (called as parameters of second of kind) of the association scheme are given respectively: $v = 10m$, $n_1 = m-1$, $n_2 = 6m$, $n_3 = 3m$, and

$$P_1 = \begin{bmatrix} m-2 & 0 & 0 \\ 0 & 6m & 0 \\ 0 & 0 & 3m \end{bmatrix}, P_2 = \begin{bmatrix} 0 & m-1 & 0 \\ m-1 & 3m & 2m \\ 0 & 2m & m \end{bmatrix}$$

and $P_3 = \begin{bmatrix} 0 & 0 & m-1 \\ 0 & 4m & 2m \\ m-1 & 2m & 0 \end{bmatrix}$.

Illustration 2.1: Let $v = 30\ (= 10 \times 3)$ treatments are arranged on the vertices of a Star Polygon graph such that each vertex contains exactly three distinct treatments as shown in Fig. 1.

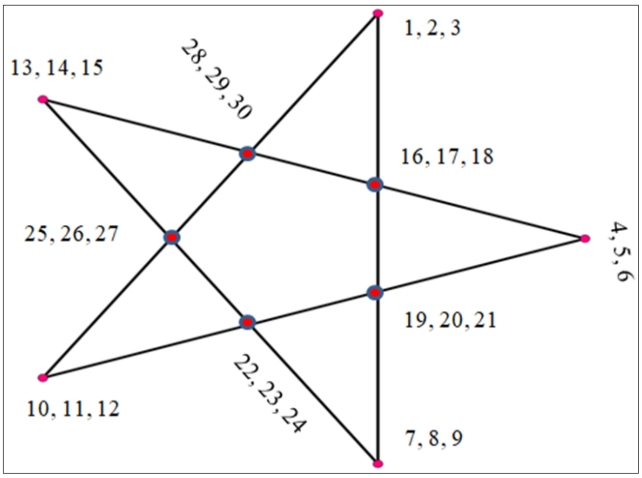

Fig 1: Arrangement of 30 treatments on vertices of a Star Polygon

Here, $n_1 = 2$, $n_2 = 18$, $n_3 = 9$ and the three associates of treatments, say 1, 2, 7 and 22 are as given in Table 1. Association matrices of this illustration are as follows:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 9 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 9 & 6 \\ 0 & 6 & 3 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 12 & 6 \\ 2 & 6 & 0 \end{bmatrix}.$$

Table 1: Different associates of treatments 1, 2, 7 and 22

Treatment	1st associates	2nd associates	3rd associates
1	2, 3	7, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 30	4, 5, 6, 13, 14, 15, 22, 23, 24
2	1, 3	7, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 30	4, 5, 6, 13, 14, 15, 22, 23, 24
7	8, 9	1, 2, 3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27	4, 5, 6, 10, 11, 12, 28, 29, 30
22	23, 24	4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 21, 25, 26, 27	1, 2, 3, 16, 17, 18, 28, 29, 30

2.1 Methods of construction of PBIB (3) designs Using Star Polygon graph

In this section, we give two construction methods of PBIB designs based on Star Polygon association scheme with three associate classes in two and three replications.

Method 2.1.1: Let $v = 10m$ $(m \geq 2)$ treatments are arranged on the vertices of a Star Polygon graph as indicated in the association scheme. By taking all possible triangles as blocks such that treatments situated on the three vertices of each triangle taken together as a block, this process implies a PBIB(3) design based on Star Polygon association scheme with parameters $v = 10m$, $b = 10$, $r = 3$, $k = 3m$, $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$.

Example 2.1.1: Let $v = 30$ treatments are arranged on the vertices of a Star Polygon graph as given in Fig. 1. By following the procedure of Method 2.1.1, we can get a PBIB (3) design based on Star Polygon association scheme with block contents as

(1, 2, 3, 16, 17, 18, 28, 29, 30)
(4, 5, 6, 16, 17, 18, 19, 20, 21)
(7, 8, 9, 19, 20, 21, 22, 23, 24)
(10, 11, 12, 22, 23, 24, 25, 26, 27)
(13, 14, 15, 25, 26, 27, 28, 29, 30)
(1, 2, 3, 7, 8, 9, 25, 26, 27)
(1, 2, 3, 10, 11, 12, 19, 20, 21)
(4, 5, 6, 13, 14, 15, 22, 23, 24)
(4, 5, 6, 10, 11, 12, 28, 29, 30)
(7, 8, 9, 13, 14, 15, 16, 17, 18)

The parameters of the above design are $v = 30$, $b = 10$, $r = 3$, $k = 9$, $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$.
A total of 9 PBIB(3) designs for $v \leq 100$ generated by Method 2.1.1 are catalogued in Table 2.1 of the Appendix along with their average variance $(\bar{V})$ and efficiency (E) as compared to a randomized complete block design. All the designs have efficiency greater than 0.8000 but less than 0.9628.

Method 2.1.2: Let $v = 10m$ $(m \geq 2)$ treatments are arranged on the vertices of a Star Polygon graph as defined in the association scheme. Form the blocks of the design each one corresponding to a quadruplets by taking together the treatments that lie on the four vertices (points) of each quadruplet as the block, which results the PBIB(3) design based on Star Polygon association scheme with parameters $v = 10m$, $b = 5$, $r = 2$, $k = 4m$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$.

Example 2.1.2: Let $v = 30$ treatments are defined on the Star Polygon association scheme. Then by using the procedure of Method 2.1.2, we can get a PBIB (3) design based on Star Polygon association scheme with block contents as

(1, 2, 3, 10, 11, 12, 25, 26, 27, 28, 29, 30)
(1, 2, 3, 7, 8, 9, 16, 17, 18, 19, 20, 21)
(4, 5, 6, 10, 11, 12, 19, 20, 21, 22, 23, 24)
(4, 5, 6, 13, 14, 15, 16, 17, 18, 28, 29, 30)
(7, 8, 9, 13, 14, 15, 22, 23, 24, 25, 26, 27)

The parameters of this design are $v = 30$, $b = 5$, $r = 2$, $k = 12$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$.
A total of 9 PBIB(3) designs generated by Method 2.1.2 are catalogued in Table 2.2 of the Appendix for $v \leq 100$. In Table 2.2, efficiency of the designs constructed using this method ranges from 0.8878 to 0.9763.

Remark 2.1: For $m = 1$, the Method 2.1.1 and Method 2.1.2 are reduced to 2-class PBIB design from Theorem 3.1 and Theorem 3.2 respectively, further details on this, reader may refer to Garg and Farooq (2014) [6].

2.2 Pappus Hexagon Association scheme
Let $v = 9m$ $(m \geq 2)$ be the number of treatments. Arrange these treatments on a Pappus Hexagon graph's vertices so that each vertex has exactly m distinct treatments. The association scheme is now defined as follows: Two treatments are first associates if they lie on same vertex. Second associates, if they lie on two different vertices of same triangle that intersect the vertex of first associates and third associates, otherwise. The parameters and association matrices of the association scheme are given respectively: $v = 9m$, $n_1 = m - 1$, $n_2 = 6m$, $n_3 = 2m$, and

$$P_1 = \begin{bmatrix} m-2 & 0 & 0 \\ 0 & 6m & 0 \\ 0 & 0 & 2m \end{bmatrix}, P_2 = \begin{bmatrix} 0 & m-1 & 0 \\ m-1 & 3m & 2m \\ 0 & 2m & 0 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 0 & m-1 \\ 0 & 6m & 0 \\ m-1 & 0 & m \end{bmatrix}.$$

Illustration 2.2: Let $v = 18\ (= 9 \times 2)$ treatments are arranged on the vertices of a Pappus Hexagon graph such that each vertex contains exactly two distinct treatments is given in Fig. 2.

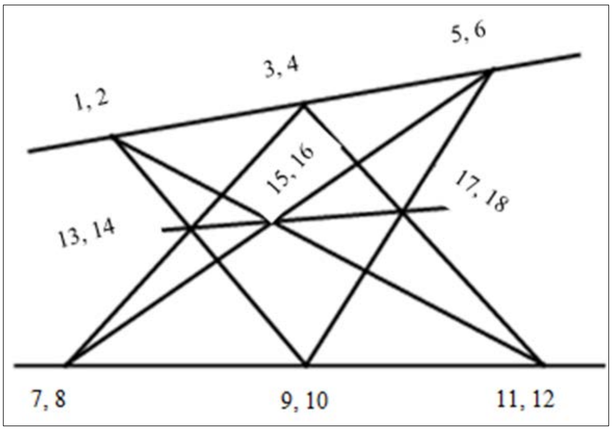

Fig 2: Arrangement of 20 treatments on vertices of a Pappus Hexagon

Here, $n_1 = 1$, $n_2 = 12$, $n_3 = 4$ and the three associates of treatments, say 1, 2, 3 and 7 are as given in Table 2. Association matrices of above illustration are:

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 4 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 6 & 4 \\ 0 & 4 & 0 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 12 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Table 2: Different associates of treatments 1, 2, 3 and 7

Treatment	1st associates	2nd associates	3rd associates
1	2	3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16	7, 8, 17, 18
2	1	3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16	7, 8, 17, 18
3	4	1, 2, 5, 6, 7, 8, 11, 12, 13, 14, 17, 18	9, 10, 15, 16
7	8	3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16	1, 2, 17, 18

2.2. Method of constructing PBIB (3) design Using Pappus Hexagon graph
In this section, we give a construction method of PBIB design based on Pappus Hexagon association scheme with three associate classes in six replications.

Method 2.2.1: Let $v = 9m$ $(m \geq 2)$ treatments are defined on the Pappus Hexagon association scheme. By taking together the treatments that lies on all the three vertices of each triangle taken as the block of size $3m$. Repeat this process to all possible triangles exist in the Pappus Hexagon yields a PBIB(3) design based on Pappus Hexagon association scheme with parameters as $v = 9m$, $b = 18$, $r = 6$, $k = 3m$, $\lambda_1 = 6$, $\lambda_2 = 2$, $\lambda_3 = 0$.

Example 2.2.1: Let $v = 18$ treatments are defined on the Pappus Hexagon association scheme. Now as per the procedure of Method 2.2.1, one can get a PBIB(3) design based on Pappus Hexagon association scheme with block structure as

(1, 2, 5, 6, 15, 16); (3, 4, 7, 8, 11, 12)
(1, 2, 3, 4, 13, 14); (5, 6, 15, 16, 17, 18)
(1, 2, 13, 14, 15, 16); (5, 6, 7, 8, 9, 10)
(1, 2, 5, 6, 9, 10); (7, 8, 13, 14, 15, 16)
(1, 2, 9, 10, 11, 12); (7, 8, 9, 10, 13, 14)
(1, 2, 3, 4, 11, 12); (7, 8, 11, 12, 15, 16)
(3, 4, 13, 14, 17, 18); (9, 10, 13, 14, 17, 18)
(3, 4, 5, 6, 17, 18); (9, 10, 11, 12, 17, 18)
(3, 4, 5, 8, 11, 12); (11, 12, 15, 16, 17, 18)

The parameters of above design are $v = 18$, $b = 18$, $r = 6$, $k = 6$, $\lambda_1 = 6$, $\lambda_2 = 2$, $\lambda_3 = 0$.
The PBIB (3) designs generated by Method 2.2.1 are catalogued in Table 2.3 for $v \leq 100$ in the Appendix. This method yielded a total of 10 designs, each with efficiency in the range of 0.8500 to 0.9708, as shown in Table 2.3.

Remark 2.2: For $m = 1$, the Method 2.2.1 is reduced to Theorem 3.3 which yields two class PBIB designs found in Garg and Farooq (2014) [6].
Next higher association scheme is named as Extended G_6 graph association scheme. A construction method of 4-associate class PBIB design based on this scheme is constructed with the help of G_6 graph quoted in Garg and Farooq (2014) [6] and the scheme is defined in sequel.

2.3 Extended G6 Graph Association scheme
Let the number of treatments be $v = 6m \; \forall \; m \geq 2$. Arrange these v treatments on the six vertices (points) of a G6 graph such that each vertex contains exactly m distinct treatments. Now we define the association scheme as follows: Treatment θ_2 is the first associate of θ_1, if θ_2 lies on the same vertex of θ_1; the second associate, if θ_2 lies on any of the two vertices of same triangle that intersect the vertex of θ_1; the third associate, if θ_2 lies on the collinear points (the points that lie on the same straight line or in a single line) and fourth associates, otherwise. The parameters of the association scheme are given by $v = 6m$, $n_1 = m - 1$, $n_2 = 2m$, $n_3 = 2m$, $n_4 = m$. Association matrices of the association scheme are given below:

$$P_1 = \begin{bmatrix} (m-2) & 0 & 0 & 0 \\ 0 & 2m & 0 & 0 \\ 0 & 0 & 2m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}, P_2 = \begin{bmatrix} 0 & (m-1) & 0 & 0 \\ (m-1) & 0 & m & 0 \\ 0 & m & 0 & m \\ 0 & 0 & m & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & (m-1) & 0 \\ 0 & m & 0 & m \\ (m-1) & 0 & m & 0 \\ 0 & m & 0 & 0 \end{bmatrix}$$

$$\text{and } P_4 = \begin{bmatrix} 0 & 0 & 0 & (m-1) \\ 0 & 0 & 2m & 0 \\ 0 & 2m & 0 & 0 \\ (m-1) & 0 & 0 & 0 \end{bmatrix}.$$

Illustration 2.3: Let $v = 18 \; (= 6 \times 3)$ treatments are arranged on the vertices of G6 graph such that each vertex contains exactly three distinct treatments is given in Fig. 3.

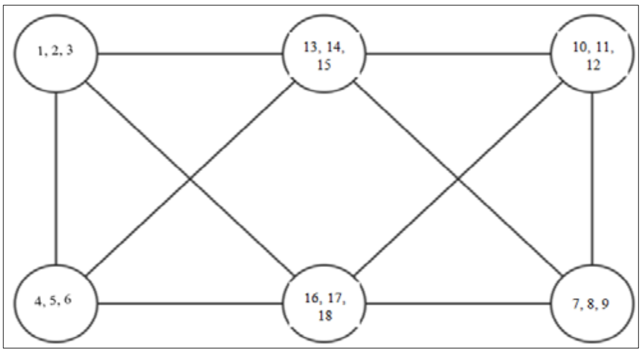

Fig 3: Arrangement of 18 treatments on vertices of G6 graph

Here, $n_1 = 2$, $n_2 = 6$, $n_3 = 6$, $n_4 = 3$ and the four associates of treatments, say 1, 2, 4, 10 and 7 are as given in Table 3. Also, association matrices of this scheme are as follows;

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} \text{ and } P_4 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 6 & 0 \\ 0 & 6 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

Table 3: Different associates of treatments 1, 2, 4, 10 and 7

Treatment	1st associates	2nd associates	3rd associates	4th associates
1	2, 3	4, 5, 6, 16, 17, 18	10, 11, 12, 13, 14, 15	7, 8, 9
2	1, 3	4, 5, 6, 16, 17, 18	10, 11, 12, 13, 14, 15	7, 8, 9
4	5, 6	1, 2, 3, 13, 14, 15	7, 8, 9, 16, 17, 18	10, 11, 12
10	11, 12	7, 8, 9, 16, 17, 18	1, 2, 3, 13, 14, 15	4, 5, 6
7	8, 9	10, 11, 12, 13, 14, 15	4, 5, 6, 16, 17, 18	1, 2, 3

2.3 Method of constructing PBIB (4) design Using G6 graph
In this section, we give a construction method of PBIB design based on Extended G6 graph association scheme with four associate classes in three replications.

Method 2.3.1: Let $v = 6m \; (m \geq 2)$ treatments are defined on Extended G6 graph association scheme. By taking together the treatments that lies on all the three vertices of each triangle taken as the block of size $3m$. Repeat this process to all possible triangles exist in the G6 graph then resultant design is the PBIB(4) design based on Extended G6 graph association scheme with parameters $v = 6m$, $b = 6$, $r = 3$, $k = 3m$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, $\lambda_4 = 0$.

Example 2.3.1: Let $v = 18$ treatments are defined on the Extended G6 graph association scheme. By using the procedure of Method 2.3.1, we can get a PBIB(4) design based on Extended G6 graph association scheme with block layout as

(1, 2, 3, 4, 5, 6, 16, 17, 18)
(1, 2, 3, 4, 5, 6, 13, 14, 15)
(1, 2, 3, 10, 11, 12, 16, 17, 18)
(7, 8, 9, 10, 11, 12, 16, 17, 18)
(4, 5, 6, 7, 8, 9, 13, 14, 15)
(7, 8, 9, 10, 11, 12, 13, 14, 15)

The parameters of the design are $v = 18$, $b = 6$, $r = 3$, $k = 9$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, $\lambda_4 = 0$.
A total of 15 PBIB (4) designs generated by Method 2.3.1 for $v \leq 100$ are catalogued in Table 2.4 of the Appendix. Efficiency of the designs from this method lies in the range of 0.8644 to 0.9821, as mentioned in Table 2.4.

Remark 2.3: For $m = 1$, the Method 2.3.1 is coincides with 3-class PBIB designs of Theorem 3.6 by Garg and Farooq (2014) [6].

3. Concluding Remarks and Future Scope
Construction of PBIB designs using graphs is simple for a given m distinct treatments with minimum number of replications and it will be the most appropriate choice for experimental situations where experimenters are constrained of resources. It can be seen that efficiency is quite high (more than 80 percent) for these designs generated by all the methods of construction. These designs can be beneficial in varietal trials in the field of agriculture where a large number of cultivars are being tested. Several authors such as Hinkelmann and Kempthorne (1963) [8]; Fyfe and Gilbert

(1963) [5]; Narain *et al*. (1974) [17]; Arya and Narain (1978) [2]; Agrawal (1985) [1]; Kaushik and Puri (1989) [11]; Singh and Hinkelmann (1995) [27]; Kaushik (1999) [12] have used various association schemes for the construction of partial diallel crosses. Hence, the proposed association schemes can also be useful to create effective partial diallel cross plans in plant and/or animal breeding experiments. Further, one may take up research on finding applications of these schemes in construction of p-rep designs [see e.g. Williams *et al*. (2011) [35]].

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Appendix

Table 2.1: PBIB (3) Designs based on Star Polygon Association Scheme with $v \leq 100$ Using Method 2.1.1

SI. No.	v	b	r	k	λ_1	λ_2	λ_3	n_1	n_2	n_3	$\bar{V}$	E
1	20	10	3	6	3	1	0	1	12	6	0.8008	0.8324
2	30	10	3	9	3	1	0	2	18	9	0.7545	0.8834
3	40	10	3	12	3	1	0	3	24	12	0.7320	0.9106
4	50	10	3	15	3	1	0	4	30	15	0.7187	0.9273
5	60	10	3	18	3	1	0	5	36	18	0.7038	0.9391
6	70	10	3	21	3	1	0	6	42	21	0.7036	0.9474
7	80	10	3	24	3	1	0	7	48	24	0.6989	0.9538
8	90	10	3	27	3	1	0	8	54	27	0.6953	0.9587
9	100	10	3	30	3	1	0	9	60	30	0.6924	0.9628

Table 2.2: PBIB (3) Designs based on Star Polygon Association Scheme with $v \leq 100$ Using Method 2.1.2

SI. No.	v	b	r	k	λ_1	λ_2	λ_3	n_1	n_2	n_3	$\bar{V}$	E
1	20	5	2	8	2	1	0	1	12	6	1.1263	0.8878
2	30	5	2	12	2	1	0	2	18	9	1.0827	0.9235
3	40	5	2	16	2	1	0	3	24	12	1.0615	0.9420
4	50	5	2	20	2	1	0	4	30	15	1.0489	0.9533
5	60	5	2	24	2	1	0	5	36	18	1.0407	0.9609
6	70	5	2	28	2	1	0	6	42	21	1.0348	0.9663
7	80	5	2	32	2	1	0	7	48	24	1.0304	0.9705
8	90	5	2	36	2	1	0	8	54	27	1.0269	0.9737
9	100	5	2	40	2	1	0	9	60	30	1.0242	0.9763

Table 2.3: PBIB (3) Designs based on Pappus Hexagon Association Scheme with $v \leq 100$ Using Method 2.2.1

SI. No.	v	b	r	k	λ_1	λ_2	λ_3	n_1	n_2	n_3	$\bar{V}$	E
1	18	18	6	6	6	2	0	1	12	4	0.3921	0.8500
2	27	18	6	9	6	2	0	2	18	6	0.3718	0.8965
3	36	18	6	12	6	2	0	3	24	8	0.3619	0.9211
4	45	18	6	15	6	2	0	4	30	10	0.3561	0.9362
5	54	18	6	18	6	2	0	5	36	12	0.3522	0.9464
6	63	18	6	21	6	2	0	6	42	14	0.3495	0.9538
7	72	18	6	24	6	2	0	7	48	16	0.3474	0.9594
8	81	18	6	27	6	2	0	8	54	18	0.3458	0.9638
9	90	18	6	30	6	2	0	9	60	20	0.3446	0.9673
10	99	18	6	33	6	2	0	10	66	22	0.3435	0.9708

Table 2.4: PBIB (4) Designs based on Extended G_6 graph Association Scheme with $v \leq 100$ Using Method 2.3.1

SI. No.	v	b	r	k	λ_1	λ_2	λ_3	λ_4	n_1	n_2	n_3	n_4	$\bar{V}$	E
1	12	6	3	6	3	2	1	0	1	4	4	2	0.7712	0.8644
2	18	6	3	9	3	2	1	0	2	6	6	3	0.7343	0.9078
3	24	6	3	12	3	2	1	0	3	8	8	4	0.7166	0.9302
4	30	6	3	15	3	2	1	0	4	10	10	5	0.7063	0.9438
5	36	6	3	18	3	2	1	0	5	12	12	6	0.6995	0.9530
6	42	6	3	21	3	2	1	0	6	14	14	7	0.6947	0.9596
7	48	6	3	24	3	2	1	0	7	16	16	8	0.6911	0.9645
8	54	6	3	27	3	2	1	0	8	18	18	9	0.6883	0.9684
9	60	6	3	30	3	2	1	0	9	20	20	10	0.6861	0.9716
10	66	6	3	33	3	2	1	0	10	22	22	11	0.6843	0.9741
11	72	6	3	36	3	2	1	0	11	24	24	12	0.6828	0.9762
12	78	6	3	39	3	2	1	0	12	26	26	13	0.6816	0.9780
13	84	6	3	42	3	2	1	0	13	28	28	14	0.6805	0.9796
14	90	6	3	45	3	2	1	0	14	30	30	15	0.6795	0.9809
15	96	6	3	48	3	2	1	0	15	32	32	16	0.6787	0.9821